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# Reductions of (2 + 1)-dimensional integrable systems via mixed potential–eigenfunction constraints

Boris Konopelchenko†§ and Walter Strampp‡

† Service de Physique Théorique, CEN Saclay, 91191 Gif-sur-Yvette, France
‡ Fachbereich 17–Mathematik/Informatik, GH–Universität Kassel, Holländische Strasse 36, 3500 Kassel, Federal Republic of Germany

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**Abstract.** New types of reductions of $(2+1)$-dimensional integrable systems which are associated with mixed potential–eigenfunction constraints are discussed. Necessary and sufficient conditions for the admissibility of such constraints are presented. Classification results for integrable equations will be used for applying these conditions. Several typical examples are considered, including the KP, $m$KP, $(2+1)$-dimensional Savada–Kotera–Kaup–Kupershmidt, Harry Dym and Nizhnik–Veselov–Novikov equations.

## 1. Introduction

It is well known that by using constraints integrable systems can be reduced to systems containing a lower number of unknown functions or independent variables. In many cases reductions lead to interesting new integrable systems (see e.g. [1–4]). An important class of reductions is associated with constraints which include both potentials obeying non-linear integrable systems and eigenfunctions of the corresponding linear problems. For $(1+1)$-dimensional integrable systems reductions leading to finite-dimensional integrable systems have been widely discussed (see e.g. [5–10]).

A generalization of such types of reductions to the $(2+1)$-dimensional case has been proposed recently [11–17, 30], where symmetry generators have been constrained to particular symmetries. Symmetry reductions of the KP-hierarchy [11, 13, 16] and 2DTL-hierarchy [16] and other systems [12, 14, 15, 17, 30] have been considered. It has been shown that mixed potential–eigenfunction constraints reduce $(2+1)$-dimensional integrable systems to $(1+1)$-dimensional integrable systems. In the present paper we will discuss general properties of reductions of $(2+1)$-dimensional integrable systems and methods for obtaining suitable constraints.

Let us consider the $(2+1)$-dimensional integrable system

$$F(u, u_x, u_y, u_t, \ldots) = 0 \tag{1.1}$$

which is representable as the compatibility condition of the linear system for the eigenfunctions $\psi$

$$L_1(u, \partial)\psi = 0 \tag{1.2}$$

$$L_2(u, \partial)\psi = 0 \tag{1.3}$$

§ Permanent address: Institute of Nuclear Physics, Novosibirsk 90, 630090 Russia.

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where $u$ is a multicomponent vector and $L_1$ and $L_2$ are linear partial differential operators. The coefficients of $L_1$ and $L_2$ are parameterized by $u$. Let us introduce the formally adjoint system of (1.2), (1.3)

$$L_1^*(u,\partial)\psi^* = 0 \tag{1.4}$$

$$L_2^*(u,\partial)\psi^* = 0 \tag{1.5}$$

with the adjoint eigenfunctions $\psi^*$. The compatibility condition between (1.4) and (1.5) is again given by (1.1). Formulae (1.1)–(1.5) are typical representations of $(2+1)$-dimensional systems integrable by the IST method (see e.g. [1–3]). Note that the system (1.2), (1.3) or (1.4), (1.5) may not be compatible for any $u$ which obeys (1.1).

We will consider constraints on $u, \psi, \psi^*$ of the form

$$f(u, u_x, u_y, \ldots, \psi, \psi^*, \psi_x, \psi_x^*, \psi_y, \psi_y^*, \ldots) = 0 \tag{1.6}$$

where $f$ is some vector function. The constraint (1.6) is said admissable if (1.6) is compatible with (1.1)–(1.5). Admissable constraints lead to reductions of the system (1.1)–(1.5), i.e. to conversions into systems with a lower number of unknown functions or independent variables. In the cases where the function $f$ depends only on $u$ or $\psi, \psi^*$ separately, we have pure potential or pure eigenfunction reductions which have been widely discussed (see e.g. [1–4]). Here we will consider the mixed case when the constraint (1.6) contains both the potentials $u$ and the eigenfunctions $\psi$ and $\psi^*$.

Necessary and sufficient conditions are found for a subclass of such reductions (1.6). Several typical examples are considered as illustrations. They include the Kadomtsev–Petviashvili (KP), $m$KP, $(2+1)$-dimensional Savada–Kotera–Kaup–Kupershmidt, Harry Dym, Nizhnik–Veselov–Novikov and some other equations.

## 2. General approach

For the sake of simplicity we will restrict ourselves to the evolutionary case when the system (1.1), (1.5) takes the form

$$u_t = F(u, u_x, u_y, \ldots) \tag{2.1}$$

$$\psi_y = A(u, \partial_x)\psi \tag{2.2}$$

$$\psi_t = B(u, \partial_x, \partial_y)\psi \tag{2.3}$$

$$\psi_y^* = -A^*(u, \partial_x)\psi^* \tag{2.4}$$

$$\psi_t^* = -B^*(u, \partial_x, \partial_y)\psi^* \tag{2.5}$$

where $A$ and $B$ are linear operators with formally adjoints $A^*$ and $B^*$. Furthermore, the constraint (1.6) will be assumed to take the form

$$u = f(\psi, \psi^*, \psi_x, \psi_x^*, \ldots)\,. \tag{2.6}$$

Let us now discuss conditions for the admissibility of the constraint (2.6), i.e. the compatibility of (2.6) with equations (2.1)–(2.5). A necessary condition is given by

$$\left(F(f, f_x, f_y, \ldots) - \frac{\partial}{\partial t} f(\psi, \psi^*, \psi_x, \psi_x^*, \ldots)\right)_{(2.2)-(2.5),\,(2.6)} = 0\,. \tag{2.7}$$

A sufficient condition (S) is given by: the constraint (2.6) is admissable if (2.1) is identically satisfied in virtue of (2.2)–(2.5) where $u$ is given by (2.6).

Under a constraint (2.6) the system (2.2)–(2.5) is reduced to a system of equations for $\psi, \psi^*$ only. Indeed, using (2.2), (2.4) and (2.6) we can eliminate $u, u_x, u_y, \psi_y, \psi_y^*$ from (2.3) and (2.5). As a result we get the following pair of systems:

$$\psi_y = \tilde{A}(\psi, \psi^*, \psi_x, \psi_x^*, \ldots, \partial_x)\psi \tag{2.8}$$

$$\psi_y^* = -\tilde{A}^*(\psi, \psi^*, \psi_x, \psi_x^*, \ldots, \partial_x)\psi^* \tag{2.9}$$

and

$$\psi_t = \tilde{B}(\psi, \psi^*, \psi_x, \psi_x^*, \ldots, \partial_x)\psi \tag{2.10}$$

$$\psi_t^* = -\tilde{B}^*(\psi, \psi^*, \psi_x, \psi_x^*, \ldots, \partial_x)\psi^* \tag{2.11}$$

where the operators $\tilde{A}, \tilde{A}^*$ and $\tilde{B}, \tilde{B}^*$ are obtained from $A, A^*$ and $B, B^*$ through elimination of $u, u_x, u_y, \psi_y, \psi_y^*$. If the constraint (2.6) is admissable, then the *reduced system (2.8)–(2.11) is compatible, i.e. system (2.8), (2.9) together with the* system (2.10), (2.11) form a pair of commuting flows. This implies the following necessary condition (N) for admissibility: the constraint (2.6) is admissable if the system (2.8), (2.9) possesses the system (2.10), (2.11) as a higher-order Lie–Bäcklund symmetry.

Usually a whole hierarchy of integrable equations, i.e. Lie–Bäcklund symmetries, is associated with a given integrable equation (see [1–3]). In the case of (2.1) a hierarchy of Lie–Bäcklund symmetries is obtained as compatibility conditions between the linear problem (2.2) and (2.3) (or (2.4) and (2.5)) where $t \to t_n$ and $B \to B_n$, $B^* \to B_n^*$ with higher-order differential operators $B_n$, $B_n^*$. Imposing the constraint (2.6) on those systems and repeating the arguments given above we arrive at the higher-order analogue of the system (2.8), (2.9)

$$\psi_{t_n} = \tilde{B}_n(\psi, \psi^*, \psi_x, \psi_x^*, \ldots, \partial_x)\psi \tag{2.12}$$

$$\psi_{t_n}^* = -\tilde{B}_n^*(\psi, \psi^*, \psi_x, \psi_x^*, \ldots, \partial_x)\psi^* \tag{2.13}$$

where $n = 1, 2, 3 \ldots$. All systems (2.12), (2.13) commute with the system (2.8), (2.9). So, applying the constraint (2.6) to the whole $(2+1)$-dimensional hierarchy we obtain an infinite family of commuting flows. As a consequence, one can reformulate the necessary condition (N) for admissibility as follows: the constraint (2.6) is admissable if the system (2.8), (2.9) possesses the systems (2.12), (2.13) as higher-order Lie–Bäcklund symmetries.

If we take the sufficient condition (S) into account together with the fact that (2.1) is obtained as the compatibility condition between (2.2) and (2.3) or (2.4) and (2.5) then the necessary condition (N) turns out to be a necessary and sufficient one. The existence of classification results for the second- and third-order integrable equations and systems (see e.g. [18–23]) will allow us to use these criteria rather effectively. As a necessary condition we first check with the the help of a list of integrable systems whether the system (2.8), (2.9) forms an integrable system, i.e. a system which possesses an infinite number of Lie–Bäcklund symmetries. Then we ckeck whether the system (2.10), (2.11) represents a symmetry of (2.8), (2.9). If both conditions are satisfied then the constraint (2.6) is admissable.

We emphasize also that the systems (2.8), (2.9) and (2.10), (2.11) provide a natural decomposition of the $(2+1)$-dimensional integrable equation (2.1). Namely, if the functions $\psi$ and $\psi^*$ obey the $(1+1)$-dimensional systems (2.8), (2.9) and (2.10), (2.11) then $u$ given by (2.6) provides a solution of the $(2+1)$-dimensional equation (2.1). This allows us to construct wide classes of solutions of (2.1) using the common solutions of the systems (2.8), (2.9) and (2.10), (2.11) (see e.g. [13]).

## 3. Third-order integrable equations

Let us consider $(2+1)$-dimensional systems integrable by second-order linear problems (2.2). Our first example is given by the system (see e.g. [24])

$$v_{0t} = -6v_0v_{0x} - 2(v_0v_1)_y + 4v_{0xy} + 2v_0v_{1x} + 2v_0v_1v_{1x} + v_{0x}v_{1x} + \tfrac{1}{2}v_1^2v_{0x} + 3v_{0x}\partial_x^{-1}v_{1y} - \phi_y + v_1\phi_x - v_0\phi_{xx} \tag{3.1}$$

$$v_{1t} = 2v_{0xx} + 6v_{0y} - v_{1xxx} - 2(v_0v_1)_x + \tfrac{3}{2}v_1^2v_{1x} + 3v_{1x}\partial_x^{-1}v_{1y} - 3\partial_x^{-1}v_{1yy} - 2\phi_x \tag{3.2}$$

where $\phi(x,y,t)$ is an arbitrary function. The system (3.1), (3.2) is equivalent with the compatibility condition of the following linear system [24]:

$$L_1\psi = \psi_y + \psi_{xx} + v_1\psi_x + v_0\psi = 0 \tag{3.3}$$

$$L_2\psi = \psi_t + 4\psi_{xxx} + 6v_1\psi_{xx} + \left(3v_{1x} + \tfrac{3}{2}v_1^2 - 3\partial_x^{-1}v_{1y} + 6v_0\right)\psi_x + (v_{0x} + 2v_0v_1 - \phi)\,\psi = 0\,. \tag{3.4}$$

At $v_1 = 0,\ \phi = v_{0x} + 3\partial_x^{-1}v_{0y}$ the system (3.1), (3.2) is reduced to the KP equation for $v_0$. At $v_0 = 0,\ \phi = 0$ it is reduced to the $m$KP equation for $v_1$ see [24, 25].

We study constraints of the form

$$v_0 = f_0(\psi, \psi_x, \ldots) \tag{3.5}$$

$$v_1 = f_1(\psi, \psi_x, \ldots) \tag{3.6}$$

which we impose on (3.3), (3.4). Now, under (3.5), (3.6) equation (3.3), (3.4) become

$$\psi_y = -\psi_{xx} + C(\psi, \psi_x, \ldots) \tag{3.7}$$

$$\psi_t = -4\psi_{xxx} + D(\psi, \psi_x, \ldots) \tag{3.8}$$

where $C$ and $D$ are obtained by introducing (3.5), (3.6) into (3.3), (3.4)—for instance $C = -f_1\psi_x - f_0\psi$. According to the necessary and sufficient conditions for admissibility, (3.7) should possess an infinite set of higher-order symmetries, the first of which is given by (3.8). There exists a wide class of second-order evolution equations possessing an infinite number of symmetries (see e.g. [23]). The Burgers and the potential Burgers equations are the simplest of them.

Now we restrict ourselves to the KP case ($v_1 = 0,\ \phi = v_{0x} + 3\partial_x^{-1}v_{0y}$) and the $m$KP case ($v_0 = 0,\ \phi = 0$). In the KP case, under the constraint

$$v_0 = 2\psi_x \tag{3.9}$$

(3.7) becomes the Burgers equation $\psi_y = -\psi_{xx} - 2\psi\psi_x$, while (3.8) becomes the first higher-order Burgers equation $\frac{1}{4}\psi_t = -\psi_{xxx} - 3\psi_x^2 - 3\psi_{xx}\psi - 3\psi^2\psi_x$. This reduction has been already discussed in [30] and [17]. In the $m$KP case, under the constraint

$$v_1 = 2\psi \tag{3.10}$$

(3.7) again becomes the Burgers equation and (3.8) becomes the first higher-order Burgers equation. Next, instead of (3.10) we shall use the constraint

$$v_1 = \psi_x \,. \tag{3.11}$$

Under (3.11) equation (3.7) becomes the potential Burgers equation $\psi_y = -\psi_{xx} - \psi_x^2$. However, under (3.11) equation (3.8) does not become a symmetry of (3.7). This example shows that a constraint may convert the linear problem (3.3) into an integrable equation without being admissible.

As a further example of a non-linear $(2+1)$-dimensional equation associated with the second-order spectral problem we shall consider the Harry–Dym equation

$$u_t = -u^3 u_{xxx} - 3u^{-1}\left(u^2 \partial_x^{-1} \frac{u_y}{u^2}\right)_y \tag{3.12}$$

which is equivalent to the compatibility condition for the following system [25]:

$$\psi_y = -u^2 \psi_{xx} \tag{3.13}$$

$$\psi_t = -4u^3 \psi_{xxx} + 6u^2 \left(u_x - \partial_x^{-1} \frac{u_y}{u^2}\right) \psi_{xx} \,. \tag{3.14}$$

The simplest reduction of (3.12) and (3.13), (3.14) is due to the constraint

$$u = \psi \,. \tag{3.15}$$

Under the constraint (3.15) the linear problems (3.13), (3.14) take the form

$$\psi_y = -\psi^2 \psi_{xx} \tag{3.16}$$

$$\psi_t = -4\psi^3 \psi_{xxx} - 12\psi^2 \psi_x \psi_{xx} \,. \tag{3.17}$$

Equation (3.16) is a well known non-linear diffusion equation which is linearizable while (3.17) is the first higher-order symmetry of (3.16)—see e.g. [23].

Note that the system (3.16), (3.17) provides a decomposition of the $(2+1)$-dimensional Harry–Dym equation (3.12). If $\psi$ obeys (3.16), (3.17) then it will solve (3.12). Further reductions of the system (3.12) can be studied with the use of the results of [23].

## 4. Reductions to systems of equations

In what follows, in addition to the linear system (3.3), (3.4), we shall take into account its adjoint version

$$L_1^*\psi^* = -\psi_y^* + \psi_{xx}^* - v_1\psi_x^* + (v_0 - v_{1x})\psi^* = 0 \tag{4.1}$$

$$\begin{aligned} L_2^*\psi^* = -\psi_t^* - 4\psi_{xxx}^* + 6v_1\psi_{xx}^* + \left(-3v_{1x} - \tfrac{3}{2}v_1^2 + 3\partial_x^{-1}v_{1y} - 6v_0 + 12v_{1x}\right)\psi_x^* \\ + \left(3v_{1xx} - 3v_1v_{1x} + 3v_{1y} - 2v_{0x} + 2v_0v_1 - \phi\right)\psi^* = 0\,. \end{aligned} \tag{4.2}$$

We shall study constraints of the form

$$v_0 = f_0(\psi, \psi^*, \ldots) \tag{4.3}$$

$$v_1 = f_1(\psi, \psi^*, \ldots) \tag{4.4}$$

under which (3.3) and (4.1) become

$$\psi_y = -\psi_{xx} - C(\psi, \psi^*, \ldots) \tag{4.5}$$

$$\psi_y^* = \psi_{xx}^* + C^*(\psi, \psi^*, \ldots)\,. \tag{4.6}$$

The classification of systems of the type (4.5), (4.6) with $C = C(\psi, \psi^*, \psi_x, \psi_x^*)$ and $C = C^*(\psi, \psi^*, \psi_x, \psi_x^*)$ which possess an infinite number of non-trivial symmetries has been given in [21, 22]. So, according to the condition (N), constraints (4.3), (4.4) are candidates for admissible constraints whenever the functions

$$f_1\psi_x + f_0\psi = C(\psi, \psi^*, \psi_x, \psi_x^*) \tag{4.7}$$

$$-f_1\psi_x^* + (f_0 - f_{1,x})\psi^* = C^*(\psi, \psi^*, \psi_x, \psi_x^*) \tag{4.8}$$

belong to the list presented in [22].

For given $C$ and $C^*$ the functions $f_0$ and $f_1$ are defined by

$$(f_1\psi\psi^*)_x = C\psi^* - C^*\psi \tag{4.9}$$

$$f_0 = \frac{1}{\psi}(C - f_1\psi_x)\,. \tag{4.10}$$

So, according to the sufficient condition (S), if the system (3.1), (3.2) is satisfied in virtue of (4.5), (4.6) and the corresponding system for the $t$-flow where the functions $f_0$ and $f_1$ are determined by (4.9), (4.10) then the constraint (4.3), (4.4) is admissable. Note that this is a straightforward check.

Let us consider the simpler cases of the KP and $m$KP equations. For the KP equation $v_1 = 0$, $f_1 = 0$ and therefore

$$f_0 = \frac{1}{\psi}C \tag{4.11}$$

and

$$C\psi^* = C^*\psi\,. \tag{4.12}$$

It is not difficult to check, using the list of $C$ and $C^*$ given in [22], that the condition (4.11) selects the only possibility

$$C = \psi^2\psi^* \tag{4.13}$$

$$C^* = \psi\psi^{*2} \tag{4.14}$$

i.e.

$$f_0 = \psi\psi^* . \tag{4.15}$$

So for the KP equation we have the only possibility for an admissible reduction

$$v_0 = \psi\psi^* . \tag{4.16}$$

within the class (4.3), (4.4). Note that the quantity $\psi\psi^*$ plays the role of a symmetry generator for the KP equation and that the symmetry constraint (4.16) appears naturally in our approach.

Under the reduction (4.16) the problems (3.3)–(4.1) and (3.4)–(4.2) are converted into the AKNS:

$$\psi_y = -\psi_{xx} - \psi^2\psi^* \tag{4.17}$$

$$\psi^*_y = \psi^*_{xx} + \psi^{*2}\psi \tag{4.18}$$

and the first higher AKNS system [11–13, 30]

$$\psi_t = -\psi_{xxx} - \psi\psi^*\psi_x \tag{4.19}$$

$$\psi^*_t = -\psi^*_{xxx} - \psi\psi^*\psi^*_x . \tag{4.20}$$

For the $m$KP equation ($v_0 = 0$) we have $f_0 = 0$ and from (4.10)

$$f_1 = \frac{C}{\psi^*_x} \tag{4.21}$$

and condition (4.9) becomes

$$C^*\psi = \left(\frac{C\psi\psi^*}{\psi_x}\right)_x - C\psi^* . \tag{4.22}$$

Working out (4.22) gives

$$C^*\psi = C_x\psi\psi^* + C(\psi\psi^*)_x - C\psi\psi^*\frac{\psi_{xx}}{\psi_x} - C\psi^* . \tag{4.23}$$

Since $C = C(\psi, \psi^*, \psi_x, \psi^*_x)$ and $C^* = C^*(\psi, \psi^*, \psi_x, \psi^*_x)$, from (4.23) we obtain $C_x\psi\psi^* = C\psi\psi^*\psi_{xx}/\psi_x$, i.e. $C = c\psi_x$, where $c$ is a constant. This means that $v_1 = c$ and we have a trivial constraint. In order to find admissable reductions of the type (4.3), (4.4) one can use also the results given in [22, 23].

In [30] a reduction has been found of the modified KP equation to the derivative non-linear Schrödinger equation. This approach uses a symmetry generator for the modified KP equation which is given in terms of (3.3), (3.4) and a derivative version of (4.1), (4.2).

## 5. Fifth-order integrable systems

Now let us consider $(2+1)$-dimensional integrable equations associated with third-order linear problems. The first examples are the Savada–Kotera (SK) and the Kaup–Kupershmid (KK) equations. They are of the form

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5\alpha u_x u_{xx} + 5u^2 u_x + 5u_{xxy} - 5\partial^{-1}u_{yy} + 5uu_y + 5u_x\partial^{-1}u_y \tag{5.1}$$

where $\alpha = 1$ for the SK equation and $\alpha = \frac{5}{2}$ for the KK equation. For the SK equation the linear problem is given by [25]

$$\psi_y = -\psi_{xxx} - u\psi_x \tag{5.2}$$

$$\psi_t = -9\psi_{xxxxx} - 15u\psi_{xxx} - 15u_x\psi_{xx} - \left[10u_{xx} + 5u^2 - 5(\partial^{-1}u_y)\right]\psi_x \tag{5.3}$$

while for the KK equation it is given by [25]

$$\psi_y = -\psi_{xxx} - u\psi_x - \tfrac{1}{2}u_x\psi \tag{5.4}$$

$$\begin{aligned}\psi_t = {} & -9\psi_{xxxxx} - 15u\psi_{xxx} - \tfrac{45}{2}u_x\psi_{xx} - \left[\tfrac{35}{2}u_{xx} + 5u^2 - 5(\partial^{-1}u_y)\right]\psi_x \\ & - \left(5uu_x - \tfrac{5}{2}u_y + 5u_{xxx}\right)\psi\,.\end{aligned} \tag{5.5}$$

Here we will consider constraints of the form

$$u = f(\psi, \psi_x, \psi_{xx}, \ldots)\,. \tag{5.6}$$

Under those constraints the linear problems (5.2) and (5.4) become

$$\psi_y = -\psi_{xxx} - A(\psi, \psi_x, \ldots) \tag{5.7}$$

where

$$A_{\mathrm{SK}} = f(\psi, \psi_x, \ldots)\psi_x \tag{5.8}$$

for the SK equation and

$$A_{\mathrm{KK}} = f(\psi, \psi_x, \ldots)\psi_x + f(\psi, \psi_x, \ldots)_x\psi \tag{5.9}$$

for the KK equation.

According to the necessary condition (N) (5.7) should possess infinitely many symmetries. The classification of such equations of the form (5.7) has been given in [18–20]. It includes six essentially different cases. In the first case we have a linear equation. The next two cases correspond to the KDV and the modified KDV equations. In the fourth case we have the Calogero–Degasperis equation. The equations of the fifth case include Weierstra 3-functions and the last case essentially reduces to the Krichever–Novikov (or KDV singularity manifold) equation.

In the first case we have the trivial constraint $u = 1$ and the reduced equations corresponding to (5.2) and (5.4) become linear equations with constant coefficients. The second and the third cases lead us to the reduction of (5.2) and (5.4) to the

KDV and modified KDV equations. Let us first consider the SK equation. The above reductions are

$$u_{\mathrm{SK}} = 6\psi \tag{5.10}$$

and

$$u_{\mathrm{SK}} = 6\psi^2 . \tag{5.11}$$

Imposing the constraint (5.10) on the linear problems (5.2), (5.3) gives

$$\psi_y = -\psi_{xxx} - 6\psi\psi_x \tag{5.12}$$

$$\tfrac{1}{9}\psi_t = -\psi_{xxxxx} - 10\psi\psi_{xxx} - 20\psi_x\psi_{xx} - 30\psi^2\psi_x \tag{5.13}$$

while imposing the constraint (5.11) on those problems gives

$$\psi_y = -\psi_{xxx} - 6\psi^2\psi_x \tag{5.14}$$

$$\tfrac{1}{9}\psi_t = -\psi_{xxxxx} - 10\psi^2\psi_{xxx} - 40\psi\psi_x\psi_{xx} - 10\psi_x^3 - 30\psi^4\psi_x . \tag{5.15}$$

Equations (5.12), (5.13) and (5.14), (5.15) are the KDV and the modified KDV equations and their first higher equations. The reductions (5.10) and (5.11) for the SK equation first have been found by a different approach in [14] and [30].

The reductions which are close to (5.10) and (5.11) and which were not considered in [14] and [30] are

$$u_{\mathrm{SK}} = 6\psi_x \tag{5.16}$$

and

$$u_{\mathrm{SK}} = 6\psi_x^2 . \tag{5.17}$$

It is easy to see that (5.16) reduces (5.2), (5.3) to the potential KDV and its higher-order symmetry. The constraint (5.17) converts (5.2) into the potential $m$KDV equation. However, under (5.17), (5.3) does not become a higher-order symmetry. So the constraint (5.17) is not admissable. Note that the reductions (5.10), (5.11) and (5.16) lead to decompositions of the SK equation (5.1) into $(1+1)$-dimensional problems.

Now, let us consider the KK equation (5.1) and the reduction to the second and third cases. The constraint

$$u_{\mathrm{KK}} = 4\psi \tag{5.18}$$

reduces (5.2) to the KDV (5.12), but it is not difficult to show that the reduced version of (5.3) is different to the higher-order KDV equation (5.13) and that the KK equation (5.1) is not a consequence of these equations. So the constraint (5.18) is not admissable for the KK equation.

The next possibility due to the necessary condition (N) is the constraint

$$u_{\mathrm{KK}} = 3\psi^3 . \tag{5.19}$$

The constraint (5.19) reduces (5.2) to the $m$KDV equation (5.14), but as one can show, equation (5.3) is reduced to an equation which differs from the higher-order $m$KDV equation by the coefficient in front of the term $\psi^4\psi_x$. So (5.19) is also not admissible for the KK equation. Finally, due to the necessary condition (N) we can consider the constraint

$$u_{\mathrm{KK}} = -\frac{3}{2}\frac{\psi_{xx}^2}{\psi_x^2} \tag{5.20}$$

which converts (5.2) into the Krichever–Novikov equation [19]

$$\psi_t = -\psi_{xxx} + \frac{3}{2}\frac{\psi_{xx}^2}{\psi_x}. \tag{5.21}$$

The admissibility of (5.20) is an open problem and will be studied elsewhere. It is connected with the compatibility of systems of equations which are not of the evolutionary type.

Our next example is the following equation [25]:

$$u_t = u^{5/2}u_{xxxx} + 10u^{-1/2}\left(u^{3/2}\partial^{-1}u_y^{-1/2}\right)_y + 5uu_{xxy} - \tfrac{15}{2}u_{xy}u_x + \tfrac{15}{2}u^{-1}u_xu_y \tag{5.22}$$

which is equivalent to the compatibility condition for the system [25]

$$\psi_y = -u^{3/2}\psi_{xxx} \tag{5.23}$$

$$\begin{aligned}\psi_t &= -9u^{5/2}\psi_{xxxxx} - \tfrac{45}{2}u^{3/2}u_x\psi_{xxxx}\\ &= -\left[15u^{3/2}\partial^{-1}(u^{-1/2})_y + 15u^{\frac{3}{2}}u_{xx}\right]\psi_{xxx}.\end{aligned} \tag{5.24}$$

The simplest reduction for the system (5.23), (5.24) is

$$u = \psi. \tag{5.25}$$

Indeed, one can check that under the constraint (5.25) equations (5.23), (5.24) are reduced to the commuting system

$$\psi_y = -\psi^{3/2}\psi_{xxx} \tag{5.26}$$

$$\psi_t = -9\psi^{5/2}\psi_{xxxxx} - \tfrac{45}{2}\psi^{3/2}(\psi_x\psi_{xxx})_x \tag{5.27}$$

and (5.22) is satisfied due to (5.26), (5.27).

According to the necessary condition (N) the other possible constraint is

$$u = \psi^2. \tag{5.28}$$

Under this constraint (5.23) is reduced to the well known $(1+1)$-dimensional Harry–Dym equation

$$\psi_y + \psi^3\psi_{xxx} = 0. \tag{5.29}$$

But (5.24) does not reduce to the higher-order Harry–Dym equation. So the constraint (5.28) is not admissible.

Other possibilities can be analysed using the list of third-order non-linear $C$-integrable equations presented in [20, 22].

## 6. The Nizhnik–Veselov–Novikov equation

Our last example is the Nizhnik–Veselov–Novikov (NVN) equation [27,28]

$$u_t = -\kappa_1 u_{\xi\xi\xi} - \kappa_2 u_{\eta\eta\eta} + 3\kappa_1 \left[u\left(\partial_\xi^{-1} u_\eta\right)\right]_\eta + 3\kappa_2 \left[u\left(\partial_\eta^{-1} u_\xi\right)\right]_\xi \tag{6.1}$$

where $\partial_\xi = \partial_x - \sigma\partial_y$, $\partial_\eta = \partial_x + \sigma\partial_y$, $\sigma^2 = +1, -1$ and $\kappa_1, \kappa_2$ are arbitrary constants. Equation (6.1) is equivalent to the compatibility condition for the linear system [27,28]

$$L_1\psi = -\psi_{\xi\eta} + u\psi = 0 \tag{6.2}$$

$$L_2\psi = \psi_t + \kappa_1\psi_{\xi\xi\xi} + \kappa_2\psi_{\eta\eta\eta} - 3\kappa_1\left(\partial_\xi^{-1}u_\eta\right)\psi_\eta - 3\kappa_2\left(\partial_\eta^{-1}u_\xi\right)\psi_\xi = 0\,. \tag{6.3}$$

The operator representation of the compatibility condition for (6.1) is given by Manakov's triad equation [27,28]

$$[L_1, L_2] = -3\left(\kappa_1\left(\partial_\xi^{-1}u_{\eta\eta}\right) + \kappa_2\left(\partial_\eta^{-1}u_{\xi\xi}\right)\right)\mathrm{L}_1\,. \tag{6.4}$$

The adjoint linear system for (6.1) takes the form

$$L_1\psi^* = -\psi^*_{\xi\eta} + u\psi^* = 0 \tag{6.5}$$

$$L_2\psi^* = -\psi^*_t - \kappa_1\psi^*_{\xi\xi\xi} - \kappa_2\psi^*_{\eta\eta\eta} + 3\kappa_1\left((\partial_\xi^{-1}u_\eta)\psi^*\right)_\eta + 3\kappa_2\left((\partial_\eta^{-1}u_\xi)\psi^*\right)_\xi = 0\,. \tag{6.6}$$

The case of the NVN equation (6.1) is different from the previous ones due to the non-evolutionary character of the linear systems (6.2), (6.3) and (6.5), (6.6).

It is not difficult to see that the system (6.1), (6.2)–(6.3), (6.5)–(6.6) admits the reduction

$$u = \psi^*\,. \tag{6.7}$$

Indeed, under this constraint the systems (6.2), (6.3) and (6.5), (6.6) reduce to

$$\psi_{\xi\eta} = \psi\psi^* \tag{6.8}$$

$$\psi^*_{\xi\eta} = \psi^{*2}. \tag{6.9}$$

$$\psi_t = -\kappa_1\psi_{\xi\xi\xi} - \kappa_2\psi_{\eta\eta\eta} + 3\kappa_1\left(\partial_\xi^{-1}\psi^*_\eta\right)\psi_\eta + 3\kappa_2\left(\partial_\eta^{-1}\psi^*_\xi\right)\psi_\xi \tag{6.10}$$

$$\psi^*_t = -\kappa_1\psi^*_{\xi\xi\xi} - \kappa_2\psi^*_{\eta\eta\eta} + 3\kappa_1\left((\partial_\xi^{-1}\psi^*_\eta)\psi^*\right)_\eta + 3\kappa_2\left((\partial_\eta^{-1}\psi^*_\xi)\psi^*\right)_\xi \tag{6.11}$$

and (6.11) coincides with (6.1). The system (6.8)–(6.11) is a compatible one. Equations (6.8), (6.9) form a closed two-dimensional system.

In the particular case of $\kappa_2 = 0$ the system (6.8)–(6.11) is equivalent to the two commuting non-local systems of evolution type

$$\psi_\eta = \partial_\xi^{-1}(\psi\psi^*) \tag{6.12}$$

$$\psi_\eta^* = \partial_\xi^{-1}(\psi^{*2}) \tag{6.13}$$

$$\psi_t = -\kappa_1\psi_{\xi\xi\xi} + 3\kappa_1\left(\partial_\xi^{-1}(\psi\psi^*)\right)\partial_\xi^{-2}(\psi^{*2}) \tag{6.14}$$

$$\psi_t^* = -\kappa_1\psi_{\xi\xi\xi}^* + 3\kappa_1\partial_\xi^{-1}(\psi^{*2})\partial_\xi^{-2}(\psi^{*2}) + 6\kappa_1\psi^*\partial_\xi^{-2}\left(\psi^*\partial_\xi^{-1}(\psi^{*2})\right). \tag{6.15}$$

A similar equation is obtained in the case $\kappa_1 = 0$ with the substitution $\xi \longleftrightarrow \eta$.

Among other possible constraints of the system (6.1)–(6.3) at $\kappa_2 = 0$ which obeys the neccessary condition (N) is the following:

$$u = 2\partial_\eta^{-1}(\psi^2)_\xi. \tag{6.16}$$

Under this constraint equations (6.2), (6.3) become

$$\psi_{\xi\eta} = 2\psi\partial_\eta^{-1}(\psi^2)_\xi \tag{6.17}$$

$$\psi_t = -\kappa_1\psi_{\eta\eta\eta} + 6\kappa_1\psi^2\psi_\eta. \tag{6.18}$$

Equation (6.17) is integrable (see [29]) and (6.18) is the modified KDV. So, under (6.16) the linear problems (6.2), (6.3) are both converted into integrable equations. However, those equations turn out not to be compatible.

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